Online Appendix*
"Cheap Talk in Complex Environments"

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A Preliminaries

Brownian Motion

In the main text, we studied the problem with outcome mapping $\psi(a)$ given by:

$$\psi(a) := \psi_0 + \mu a + \sigma W(a),$$

where $W(\cdot)$ is the Wiener process over $\mathbb{R}$ with $\mu \in \mathbb{R}$ and $\psi_0, \sigma \in \mathbb{R}_+$. For conciseness, the results in the online appendix are stated in terms of the standard Brownian motion with drift μ, scale σ and initial value $X(0) = 0$ given by $X(a) := \mu a + \sigma W(a)$. We denote the set of all paths of $X(\cdot)$ by $\mathcal{X}$. Note that because $\psi(a) = X(a) + \psi_0$ all of the results about the random variables work equally well as long as all of them are scaled up by ψ_0. This means the status quo $X(0) = 0$ maps to $\psi(0) = \psi_0$, Sender's ideal point 0 maps to $-\psi_0$ and Sender's ideal point b maps to $b - \psi_0$. Doing this transformation back in our variables, the desired results can be obtained.

We are interested in $X(a)$ over the domain $[0, q] \subseteq \mathbb{R}$, for some $q \in \overline{\mathbb{R}}_+$. Properties of $X(a)$ listed below characterizes Brownian motion:

i. $X(0) = 0$, and $X(a)$ is almost surely continuous.

ii. $X(a)$ has independent increments:

   $X(a_1) - X(a_1')$ and $X(a_2) - X(a_2')$ are independent random variables for $a_1' < a_1 < a_2' < a_2$.

iii. $X(a)$ has stationary increments:

   For $a, a' \in \mathbb{R}$ the increment $X(a + a') - X(a)$ has the same distribution as $X(a')$.

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iv. The increments are normally distributed:

$$X(a+a') - X(a) \sim \mathcal{N}(\mu a', \sigma^2 a') \; \forall a, a' \in \mathbb{R}_+.$$

Stochastic Processes

Our analysis uses a variety of stochastic process that are related to Brownian motion. The processes of our interest are:

1. Brownian bridge, which is the Brownian motion with known terminal action and outcome (q, b):

$$B(a, b, q) := \{X(a) \;\mid\; X(q) = b\} \text{ for } b \in \mathbb{R} \text{ and } q \in \mathbb{R}_+.$$

2. Brownian meander, which is the Brownian motion conditioned stay above 0 over interval $[0, q]$:

$$M(a, q) := \{X(a) \;\mid\; X(a') \geq 0 \;\; \forall a' \in [0, q]\}.$$

We provide a detailed analysis of these processes in the appendix. For a more detailed discussion, the reader is referred to: Chapter 1 of Harrison (2013) and Chapter 3 of Shreve (2004) for Brownian Motion; Chapter 1.1. of Mansuy and Yor (2008) and Chapter 3.4 of Pitman and Yor (2018) for Brownian Bridges; Durrett et al. (1977), Iafrate and Orsingher (2020) and Riedel (2021) for the Brownian meander. The reader is referred to Harrison (2013) for a general treatment of the topic.

Random Variables

In our analysis, we frequently focus on random variables of Brownian motion. We will state them here in terms of $X(\cdot)$. In the main text, we apply them analogously for the process $\psi(\cdot)$. Two main random variables we are interested in are:

i. First hitting action of $x \in \mathbb{R}$: $\tau(x) := \inf\{a \in [0, q] \mid \psi(a) = x\}$.[1]

- Note that $\inf\{a \in [0, q] \mid X(a) = x\} = \inf\{a \in [0, q] \mid \psi(a) - \psi_0 = x\} = \inf\{a \in [0, q] \mid \psi(a) = x + \psi_0\}$. Thus, $\tau(x)$ maps to $\tau(x - \psi_0)$ when changing the process from $X(\cdot)$ to $\psi(\cdot)$.

ii. Infimum over the interval $[0, q]$: $\iota(q) := \inf\{\psi(a) \mid a \in [0, q]\}$.

- Similarly, $\iota(q)$ maps to $\iota(q) + \psi_0$ when changing from $X(\cdot)$ to $\psi(\cdot)$.[2]

We are also interested in the compound of the two random variables given by $\tau(\iota(q))$, more precisely:

$$\tau(\iota(q)) := \inf\{a \in [0, q] \mid X(a) \leq X(a') \; \forall a' \in [0, q]\}.$$

[1] In our model the domain of the Brownian motion are actions, but the canonical usage has the domain as the time. Thus, this random variable is frequently referred as the first hitting time.

[2] We have that: $\inf\{X(a) \mid a \in [0, q]\} = \inf\{\psi(a) + \psi(0) \mid a \in [0, q]\}$.

iii. First hitting action (time) of the minimum over $[0, q]$ $\tau_\iota(q) := \tau(\iota(q))$.

- Analogously, $\tau(\iota(q))$ maps to $\tau(\iota(q) + \psi_0)$ when changing from $X(\cdot)$ to $\psi(\cdot)$.

iv. The maximal decrease in $\psi(\cdot)$ over $[q_1, q_2] \subseteq [0, q]$: $\delta(q_1, q_2)$.

- Formally, $\delta(q_1, q_2) = \inf\{X(q_2) - X(q_1) \mid a \in [q_1, q_2]\}$. Note that this is equivalent to $\iota(q_2 - q_1)$.

B Results used in the Main Proofs

The statements here are for $X(\cdot)$ as the underlying process. The first-point strategy $m^*(\psi)$ is adapted to this setting as by shifting everything down by adding $-\psi_0$.

$$m^*(X) = \begin{cases} \tau(-x) & \text{if } \tau(-x) \leq q \\ \tau_\iota(q) & \text{if } \iota(q) > -x \end{cases}.$$

By definition of $m^*(X)$, the event $m^*(X) = r^*$ is given by the set of paths $\{X(\cdot) \in \mathcal{X} \mid \tau(-x) = r^*\} \cup \{X(\cdot) \in \mathcal{X} \mid \tau_\iota(q) = r^*, \iota(q) > -x\}$.

Note that, when going from $\psi(0)$ to $X(0)$, b is mapped to a negative number $b - \psi_0$. We frequently use the arguments of $\tau(\cdot)$ by negative variables e.g.$-x$ corresponding to $b - \psi_0$. Thus, statements such as increasing in $-x$ translate to increasing in b. Finally, we use $\phi(\cdot)$ and $\Phi(\cdot)$ to denote the standard normal probability density function and cumulative density function, respectively.[3]

B.1 Supporting Results for Lemma 4

Lemma B.1 $\lim_{r^* \to 0^+} \mathbb{P}(\tau(-x) \in dr^* \mid m^*(X) = r^*) = 0 \ \forall x, r^* \in \mathbb{R}_+$.

Proof. We will show the sufficient result that the likelihood ratio of $\tau(-x)$ and $\tau_\iota(q)$ converges to 0. This result can be shown by using the property that Lèvy processes are continuous in probability. By definition of continuity in probability of $\psi(\cdot)$, for any $\varepsilon > 0$ and $a \geq 0$ it holds that $\lim_{h \to 0} P\left(|\psi(a+h) - \psi(a)| > \varepsilon\right) = 0$. We provide a detailed proof and layout all related distributions in closed form for the interested reader.

In equation (C.10), we have the closed expressions for $\mathbb{P}(\tau(-x) \in dr^* \mid m^*(X) = r^*)$ and $\mathbb{P}(\tau_\iota(q) \in dr^*, \iota(q) > -x \mid m^*(X) = r^*)$. Using the derived expressions in the equation (C.10), we state the

[3]Precisely $\phi(x)$ is defined as $\phi(x) = \frac{1}{\sqrt{2\pi}} \exp(-\frac{x^2}{2})$ and $\Phi(x)$ is defined as $\Phi(x) = \int_{-\infty}^{x} \phi(x) = \frac{1}{2}(1 + \text{erf}(\frac{x}{\sqrt{2}}))$.

likelihood ratio as:

$$\lim_{r^*\to 0^+} \frac{\mathbb{P}(\tau(-x)\in dr^* \mid m^*(X)=r^*)}{\mathbb{P}(\tau_\iota(q)\in dr^* \mid m^*(X)=r^*)} = \lim_{r^*\to 0} \frac{\frac{x}{\sigma r^*\sqrt{r^*}}\phi\left(\frac{-x-\mu r^*}{\sigma\sqrt{r^*}}\right)}{\int_{-x}^{0} \frac{-2z}{\sigma^2 r^*\sqrt{r^*}}\phi\left(\frac{z-\mu r^*}{\sigma\sqrt{r^*}}\right)\left(\frac{\mu}{\sigma}\Phi\left(\frac{\mu(q-r^*)}{\sigma\sqrt{q-r^*}}\right)+\frac{\phi\left(\frac{-\mu(q-r^*)}{\sigma\sqrt{q-r^*}}\right)}{\sqrt{q-r^*}}\right)dz}.$$

Taking the constant out from the integral, the limit can be stated as:

$$\lim_{r^*\to 0^+} \frac{\frac{x}{\sigma r^*\sqrt{r^*}}\phi\left(\frac{-x-\mu r^*}{\sigma\sqrt{r^*}}\right)}{\left(\frac{\mu}{\sigma}\Phi\left(\frac{\mu(q-r^*)}{\sigma\sqrt{q-r^*}}\right)+\frac{\phi\left(\frac{-\mu(q-r^*)}{\sigma\sqrt{q-r^*}}\right)}{\sqrt{q-r^*}}\right)\int_{-x}^{0} \frac{-2z}{\sigma^2 r^*\sqrt{r^*}}\phi\left(\frac{z-\mu r^*}{\sigma\sqrt{r^*}}\right)dz}.$$

We can rewrite the limit as the product of two expressions, given that an unambiguous limit exists for both expressions:

$$\lim_{r^*\to 0^+} \frac{1}{\frac{\mu}{\sigma}\Phi\left(\frac{\mu(q-r^*)}{\sigma\sqrt{q-r^*}}\right)+\frac{\phi\left(\frac{-\mu(q-r^*)}{\sigma\sqrt{q-r^*}}\right)}{\sqrt{q-r^*}}} \lim_{r^*\to 0^+} \frac{\frac{x}{\sigma r^*\sqrt{r^*}}\phi\left(\frac{-x-\mu r^*}{\sigma\sqrt{r^*}}\right)}{\int_{-x}^{0} \frac{-2z}{\sigma^2 r^*\sqrt{r^*}}\phi\left(\frac{z-\mu r^*}{\sigma\sqrt{r^*}}\right)dz}. \tag{B.1}$$

We calculate the integral in the second limit term explicitly as:

$$\int_{-x}^{0} \frac{-2z}{\sigma^2 r^*\sqrt{r^*}}\phi\left(\frac{z-\mu r^*}{\sigma\sqrt{r^*}}\right)dz = \frac{2\mu}{\sigma}\left(\Phi\left(\frac{\mu\sqrt{r^*}}{\sigma}\right)-\Phi\left(\frac{\mu r^*+x}{\sigma\sqrt{r^*}}\right)\right)+2\frac{\phi\left(\frac{\mu\sqrt{r^*}}{\sigma}\right)-\phi\left(\frac{\mu r^*+x}{\sigma\sqrt{r^*}}\right)}{\sqrt{r^*}}. \tag{B.2}$$

Thus, the limit of interest can be written using (B.1) and (B.2):

$$\lim_{r^*\to 0^+} \frac{1}{\frac{\mu}{\sigma}\Phi\left(\frac{\mu(q-r^*)}{\sigma\sqrt{q-r^*}}\right)+\frac{\phi\left(\frac{-\mu(q-r^*)}{\sigma\sqrt{q-r^*}}\right)}{\sqrt{q-r^*}}} \lim_{r^*\to 0^+} \frac{\frac{x}{\sigma r^*\sqrt{r^*}}\phi\left(\frac{-x-\mu r^*}{\sigma\sqrt{r^*}}\right)}{\frac{2\mu}{\sigma}\left(\Phi\left(\frac{\mu\sqrt{r^*}}{\sigma}\right)-\Phi\left(\frac{\mu r^*+x}{\sigma\sqrt{r^*}}\right)\right)+2\frac{\phi\left(\frac{\mu\sqrt{r^*}}{\sigma}\right)-\phi\left(\frac{\mu r^*+x}{\sigma\sqrt{r^*}}\right)}{\sqrt{r^*}}}.$$

The limit of the first term exists, and can be calculated by directly plugging in $r^*=0$.

$$\lim_{r^*\to 0^+} \frac{1}{\frac{\mu}{\sigma}\Phi\left(\frac{\mu(q-r^*)}{\sigma\sqrt{q-r^*}}\right)+\frac{\phi\left(\frac{-\mu(q-r^*)}{\sigma\sqrt{q-r^*}}\right)}{\sqrt{q-r^*}}} = \frac{1}{\frac{\mu}{\sigma}\Phi\left(\frac{\mu\sqrt{q}}{\sigma}\right)+\frac{\phi\left(\frac{-\mu\sqrt{q}}{\sigma}\right)}{\sqrt{q}}}.$$

To evaluate the limit for the second expression, we distribute the limit over the the expression:

$$\frac{\lim_{r^*\to 0^+} \frac{x}{\sigma r^*\sqrt{r^*}}\phi\left(\frac{-x-\mu r^*}{\sigma\sqrt{r^*}}\right)}{\lim_{r^*\to 0^+} \frac{2\mu}{\sigma}\left(\Phi\left(\frac{\mu\sqrt{r^*}}{\sigma}\right)-\Phi\left(\frac{\mu r^*+x}{\sigma\sqrt{r^*}}\right)\right)+\lim_{r^*\to 0^+} 2\frac{\phi\left(\frac{\mu\sqrt{r^*}}{\sigma}\right)-\phi\left(\frac{\mu r^*+x}{\sigma\sqrt{r^*}}\right)}{\sqrt{r^*}}}. \tag{B.3}$$

Focusing on the limit on the numerator of equation (B.3):

$$\lim_{r^*\to 0^+} \frac{x}{\sigma r^* \sqrt{r^*}} \phi\left(\frac{-x-\mu r^*}{\sigma\sqrt{r^*}}\right) = \frac{x}{\sigma\sqrt{2\pi}} \lim_{r^*\to 0^+} \sqrt{\frac{\exp(-\frac{(-\mu r^* - x)^2}{\sigma^2 r^*})}{r^{*3}}}$$
$$= \frac{x}{\sigma\sqrt{2\pi}} \lim_{r^*\to 0^+} \frac{1}{\exp\left(\frac{(-\mu r^*-x)^2}{\sigma^2 r^*}\right)(r^*)^3} = 0.$$

The first line follows from taking the constants out. We take the square of the term in the second line for a more manageable expression. The final limit follows from taking the limit to the denominator and repeatedly applying the L'Hopital rule for limits until the r^{*3} term disappears.

The limit for first term in the denominator of equation (B.3) is given by:

$$\lim_{r^*\to 0^+} \frac{2\mu}{\sigma}\left(\Phi\left(\frac{2\mu\sqrt{r^*}}{\sigma}\right) - \Phi\left(\frac{\mu r^* + x}{\sigma\sqrt{r^*}}\right)\right) = \frac{2\mu}{\sigma}\left(\lim_{r^*\to 0^+} \Phi\left(\frac{\mu\sqrt{r^*}}{\sigma}\right) - \lim_{r^*\to 0} \Phi\left(\frac{\mu}{\sigma}\sqrt{r^*} + \frac{x}{\sigma\sqrt{r^*}}\right)\right)$$
$$= \frac{2\mu}{\sigma}(0-1) = -\frac{2\mu}{\sigma}.$$

By the definition of $\Phi(\cdot)$ we have $\lim_{r^*\to 0^+} \Phi\left(\frac{\mu\sqrt{r^*}}{\sigma}\right) = 0$ and $\lim_{r^*\to 0} \Phi\left(\frac{\mu}{\sigma}\sqrt{r^*} + \frac{x}{\sigma\sqrt{r^*}}\right) = 1$. The second term in the denominator for equation (B.3) has a limit given by:

$$\lim_{r^*\to 0^+} 2\frac{\phi\left(\frac{\mu\sqrt{r^*}}{\sigma}\right) - \phi\left(\frac{\mu r^*+x}{\sigma\sqrt{r^*}}\right)}{\sqrt{r^*}} = 2\frac{\lim_{r^*\to 0}\phi\left(\frac{\mu\sqrt{r^*}}{\sigma}\right) - \lim_{m^*\to 0}\phi\left(\frac{\mu r^*+x}{\sigma\sqrt{r^*}}\right)}{\lim_{r^*\to 0}\sqrt{r^*}} = 2\frac{\phi(0)-0}{0} = \infty.$$

Thus, the limit of interest is given by:

$$\lim_{r^*\to 0^+} \frac{\mathbb{P}(\tau(-x)\in dr^* \mid m^*(X) = r^*)}{\mathbb{P}(\tau_\iota(q)\in dr^*, \iota(q) > -x \mid m^*(X) = r^*)} = \frac{1}{\frac{\mu}{\sigma}\Phi\left(\frac{\mu\sqrt{q}}{\sigma}\right) + \frac{\phi\left(\frac{-\mu\sqrt{q}}{\sigma}\right)}{\sqrt{q}}} \left(\frac{0}{\infty - \frac{2\mu}{\sigma}}\right) = 0.$$

This concludes the proof. The likelihood ratio converges to 0, which implies that the limit of the conditional probability in the statement also converges to 0. ■

Lemma B.2 *For every $\mu \in \mathbb{R}$ and $q, \sigma^2, c \in \mathbb{R}_+$, we have that* $\lim_{a\to 0^+} \dfrac{\partial \mathbb{E}[M(a,q) \mid M(q,q) = c]}{\partial a} = \infty$.

Proof. In equation (D.14), we derive the expression for $\mathbb{E}[M(a,q) \mid M(q,q) = c]$, using the techniques developed by Riedel (2021), as:

$$\mathbb{E}[M(a,q) \mid M(q,q) = c] = \frac{\sigma^2(q-a) + \frac{c^2 a}{q}}{c} \operatorname{erf}\left[\frac{c\sqrt{a}}{\sigma\sqrt{2q(q-a)}}\right] + \exp\left(\frac{-c^2 a}{2q(q-a)\sigma^2}\right)\sqrt{\frac{2a(q-a)}{q\pi}}\sigma.$$

The derivative of this expression is given by:

$$\frac{\partial \mathbb{E}[M(a,q) \mid M(q,q)=c]}{\partial a} = \frac{\left(\frac{c^2}{q}-\sigma^2\right)\operatorname{erf}\left(\frac{c\sqrt{a}}{\sqrt{2}\sigma\sqrt{q(q-a)}}\right)}{c} + \frac{\sigma(q-2a)\exp\left(\frac{-\left(c^2 a\right)}{(2q\sigma^2(q-a))}\right)}{\sqrt{2\pi}\sqrt{q}\sqrt{a(q-a)}}$$
$$+ \frac{2\exp\left(-\frac{c^2 a}{2q\sigma^2(q-a)}\right)\left(\frac{cq\sqrt{a}}{2\sqrt{2}\sigma(q(q-a))^{\frac{3}{2}}}+\frac{c}{2\sqrt{2}\sigma\sqrt{a}\sqrt{q(q-a)}}\right)\left(\frac{c^2 a}{q}+\sigma^2(q-a)\right)}{\sqrt{\pi}c}$$
$$+ \frac{\sqrt{\frac{2}{\pi}}\sigma\sqrt{a(q-a)}\exp\left(-\frac{c^2 a}{2q\sigma^2(q-a)}\right)\left(-\frac{c^2 a}{2q\sigma^2(q-a)^2}-\frac{c^2}{2q\sigma^2(q-a)}\right)}{\sqrt{q}}.$$

We evaluate the limit using the properties of the $\operatorname{erf}(x)=\frac{2}{\sqrt{\pi}}\int_0^z e^{-t^2}\,\mathrm{d}t$ and e^{-z}.

$$\lim_{a\to 0^+}\frac{\partial \mathbb{E}[M(a,q) \mid M(q,q)=c]}{\partial a} = \underbrace{\frac{\left(\frac{c^2}{q}-\sigma^2\right)\operatorname{erf}\left(\frac{c\sqrt{a}}{\sqrt{2}\sigma\sqrt{q(q-a)}}\right)}{c}}_{\to 0 \text{ as } a\to 0^+} + \underbrace{\frac{\sigma(q-2a)\exp\left(\frac{-\left(c^2 a\right)}{(2q\sigma^2(q-a))}\right)}{\sqrt{2\pi}\sqrt{q}\sqrt{a(q-a)}}}_{\to\infty \text{ as } a\to 0^+}$$
$$+ \underbrace{\frac{2\exp\left(-\frac{c^2 a}{2q\sigma^2(q-a)}\right)\left(\frac{cq\sqrt{a}}{2\sqrt{2}\sigma(q(q-a))^{3/2}}+\frac{c}{2\sqrt{2}\sigma\sqrt{a}\sqrt{q(q-a)}}\right)\left(\frac{c^2 a}{q}+\sigma^2(q-a)\right)}{\sqrt{\pi}c}}_{\to\infty \text{ as } a\to 0^+}$$
$$+ \underbrace{\frac{\sqrt{\frac{2}{\pi}}\sigma\sqrt{a(q-a)}\exp\left(-\frac{c^2 a}{2q\sigma^2(q-a)}\right)\left(-\frac{c^2 a}{2q\sigma^2(q-a)^2}-\frac{c^2}{2q\sigma^2(q-a)}\right)}{\sqrt{q}}}_{\to 0 \text{ as } a\to 0^+}.$$

Thus, we conclude that: $\lim_{a\to 0^+}\frac{\partial \mathbb{E}[M(a,q)|M(q,q)=c]}{\partial a}=\infty$. ∎

Corollary B.1 *For every* $\mu\in\mathbb{R}$ *and* $q,\sigma^2\in\mathbb{R}_+$*, we have that* $\lim_{a\to 0^+}\mathbb{E}[M(a,q)]=\infty$.

Proof. The result follows directly by the law of iterated expectations and Lemma B.2 that characterizes the analytic expression of $\mathbb{E}[M(a,q) \mid M(q,q)=c]$. Formally, we have that

$$\begin{aligned}\frac{\partial \mathbb{E}[M(a,q) \mid M(q,q)=c]}{\partial a} &= \frac{\partial \mathbb{E}_c[\mathbb{E}[M(a,q) \mid M(q,q)=c]]}{\partial a}\\ &= \mathbb{E}_c\left[\frac{\partial \mathbb{E}[M(a,q) \mid M(q,q)=c]}{\partial a}\right].\\ \lim_{a\to 0^+}\mathbb{E}[M(a,q)] &= \mathbb{E}_c\left[\lim_{a\to 0^+}\frac{\partial \mathbb{E}[M(a,q) \mid M(q,q)=c]}{\partial a}\right]\\ &= \mathbb{E}_c\left[\infty\right]=\infty.\end{aligned}$$

The first line follows from the law of iterated expectations where $\mathbb{E}_c$ denotes expectation over values of $M(q,q)=c$, which follows from $\mathbb{P}(M(a,q)\in dc)$ derived in Equation (D.2). The second line follows by taking the derivative with respect to a inside the expectation over c and the last line follows from taking the limit of both sides and using the result in Lemma B.2, which shows that $\lim_{a\to 0^+} \frac{\partial \mathbb{E}[M(a,q)|M(q,q)=c]}{\partial a} = \infty$. ■

B.2 Supporting Results for Proposition 1

Lemma B.3 *$\mathbb{P}(\tau(-x) \in da)$ is log-concave and $\mathbb{P}(\tau(-x) \in da \mid m(\psi) = a)$ is increasing in $-x$.*

Proof. $\mathbb{P}(\tau(-x) \in da)$ is given by equation (C.7):

$$\mathbb{P}(\tau(-x) \in da) = \frac{x}{\sigma a\sqrt{a}}\phi\left(\frac{-x-\mu a}{\sigma\sqrt{a}}\right) da \quad \forall x \in \mathbb{R}_+.$$

$\phi(\cdot)$ is a log-concave function, and similarly $\frac{x}{\sigma a\sqrt{a}}$ is linear, and hence is also log-concave. The product of log-concave functions are log-concave, establishing the log-concavity of $\mathbb{P}(\tau(-x) \in da)$. For the second part of the proof, consider

$$\mathbb{P}(\tau(-x) \in da \mid m(X) = a) = \frac{\mathbb{P}(\tau(-x) \in da)}{\mathbb{P}(\tau(-x) \in da) + \mathbb{P}(\tau_\iota(q) \in da, \iota(q) > -x)}.$$

$\mathbb{P}(\tau(-x) \in da \mid m(X) = a)$ is increasing if the likelihood ratio $\frac{\mathbb{P}(\tau(-x)\in da)}{\mathbb{P}(\tau_\iota(q)\in da, \iota(q)>-x)}$ is increasing. In deriving Equation (C.11), we derive the formula for $\mathbb{P}(\tau_\iota(q) \in da, \iota(q) > -x)$ as:

$$\mathbb{P}(\tau_\iota(q) \in da, \iota(q) > -x) = 2\left(\frac{\mu}{\sigma}\Phi\left(\frac{\mu\sqrt{q-a}}{\sigma}\right) + \frac{\phi\left(\frac{\mu\sqrt{q-a}}{\sigma}\right)}{\sqrt{q-a}}\right)\int_{-x}^{0} \mathbb{P}(\tau(z) \in da)dz.$$

Thus, the likelihood ratio in equation (C.11) can be written as:

$$\frac{\mathbb{P}(\tau(-x) \in da)}{\mathbb{P}(\tau_\iota(q) \in da, \iota(q) > -x)} = \left[\frac{1}{2\left(\frac{\mu}{\sigma}\Phi\left(\frac{\mu\sqrt{q-a}}{\sigma}\right) + \frac{\phi\left(\frac{-\mu\sqrt{q-a}}{\sigma}\right)}{\sqrt{q-a}}\right)}\right]\frac{\mathbb{P}(\tau(-x) \in da)}{\int_{-x}^{0} \mathbb{P}(\tau(z) \in da)dz}.$$

The first term is readily identifiable as a positive constant. The second term is the ratio of a log-concave function evaluated at $-x$, and it is integral up to $-x$. It follows from the definition of log-concavity, as noted by Bagnoli and Bergstrom (2006), that this ratio is decreasing in x (increasing in $-x$) for a log-concave function. It is immediate, from the definition above that $\mathbb{P}(\tau(-x) \in da \mid m(X) = a)$ decreasing in x (increasing in $-x$). ■

Lemma B.4 *$-x - \mathbb{E}[\psi(r^*) \mid \tau_\iota(q) \in dm^*, \iota(q) > -x]$ is increasing in $-x$.*

Proof. The expectation $\mathbb{E}[\psi(r^*) \mid \tau_\iota(q) \in dr^*, \iota(q) > b]$ is given by

$$\frac{\int_{-x}^{0} z\mathbb{P}(\tau(z) \in da)dz}{\int_{-x}^{0} \mathbb{P}(\tau(z) \in da)dz}.$$

It directly follows from Lemma B.3 that $\mathbb{P}(\tau(z) \in da)$ is log-concave. By Bagnoli and Bergstrom (2006), we have that the expression

$$-x - \frac{\int_{-x}^{0} z\mathbb{P}(\tau(z) \in da)dz}{\int_{-x}^{0} \mathbb{P}(\tau(-z) \in da)dz}$$

is increasing in $-x$. ■

Proposition B.1 $u_R(\psi, a) = -\psi(a)^2$ *satisfies the condition given by equation* (5).

Proof. Suppose that $\psi(0) > \alpha$. For $q = q_b^{\max}$, there exists some $\tilde{a}, \tilde{r} \in \left[0, q_{\max}^b\right]$ with $\tilde{a} = \tilde{r} + a'$ and $a' > 0$ such that:

$$0 = \mathbb{E}\left[u_R(\tilde{a}) - u_R\left(\tilde{r}\right) \mid m_b^*(\psi) = \tilde{r}\right]. \tag{B.4}$$

We want to show that under quadratic preferences, the following condition is satisfied.[4]

$$\frac{\mathbb{P}\left(\tau(b) \in d\tilde{r} \mid m_b^*(\psi) = \tilde{r}\right)}{\mathbb{P}\left(\tau_\iota(q) \in d\tilde{r}, \iota(q) > b \mid m_b^*(\psi) = \tilde{r}\right)} \geq -\frac{\frac{\partial}{\partial b}\mathbb{E}\left[u_R\left(\psi(\tilde{r}) + M(a', q - \tilde{r})\right) - u_R\left(\psi(\tilde{r})\right) \mid \tau_\iota(q) = \tilde{r}, \iota(q) > b\right]}{\frac{\partial}{\partial b}\mathbb{E}\left[u_R\left(b + X(a')\right) - u_R\left(b\right)\right]}. \tag{B.5}$$

Note that, quadratic utility we have that:

$$\begin{aligned}
&\mathbb{E}\left[u_R\left(b + X(a')\right) - u_R\left(b\right)\right] = \mathbb{E}\left[-(b + X(a'))^2 + b^2\right] = -2b\mathbb{E}\left[X(a')\right] - \mathbb{E}\left[X(a')^2\right] \\
&\mathbb{E}\left[u_R\left(\psi(\tilde{r}) + M(a', q - \tilde{r})\right) - u_R\left(\psi(\tilde{r})\right) \mid \tau_\iota(q) = \tilde{r}, \iota(q) > b\right] \\
=& -2\mathbb{E}\left[\psi(\tilde{r})M(a', q - \tilde{r}) \mid \tau_\iota(q) = \tilde{r}, \iota(q) > b\right] - \mathbb{E}\left[M(a', q - \tilde{r})^2 \mid \tau_\iota(q) = \tilde{r}, \iota(q) > b\right] \\
=& -2\mathbb{E}\left[\psi(\tilde{r}) \mid \tau_\iota(q) = \tilde{r}, \iota(q) > b\right]\mathbb{E}\left[M(a', q - \tilde{r})\right] - \mathbb{E}\left[M(a', q - \tilde{r})^2 \mid \tau_\iota(q) = \tilde{r}, \iota(q) > b\right].
\end{aligned}$$

where the last line follow from the independent increments. Taking derivative with respect to b, we have that:

$$\begin{aligned}
&\frac{\partial}{\partial b}\mathbb{E}\left[u_R\left(b + X(a')\right) - u_R\left(b\right)\right] = \mathbb{E}\left[-(b + X(a'))^2 + b^2\right] = -2\mathbb{E}\left[X(a')\right] \\
&\frac{\partial}{\partial b}\mathbb{E}\left[u_R\left(\psi(\tilde{r}) + M(a', q - \tilde{r})\right) - u_R\left(\psi(\tilde{r})\right) \mid \tau_\iota(q) = \tilde{r}, \iota(q) > b\right] \\
&= -2\left(\frac{\partial}{\partial b}\mathbb{E}\left[\psi(\tilde{r}) \mid \tau_\iota(q) = \tilde{r}, \iota(q) > b\right]\right)\mathbb{E}\left[M(a', q - \tilde{r})\right].
\end{aligned}$$

[4]As discussed in the proof of Proposition 1, this condition is equivalent to condition in equation (5).

So, the condition given by equation (B.5) reduces to:

$$\frac{\mathbb{P}\left(\tau(b)\in d\tilde{r}\mid m_b^*(\psi)=\tilde{r}\right)}{\mathbb{P}\left(\tau_\iota(q)\in d\tilde{r},\iota(q)>b\mid m_b^*(\psi)=\tilde{r}\right)}\geq-\frac{\left(\frac{\partial}{\partial b}\mathbb{E}\left[\psi(\tilde{r})\mid\tau_\iota(q)=\tilde{r},\iota(q)>b\right]\right)\mathbb{E}\left[M(a',q-\tilde{r})\right]}{\mathbb{E}\left[X(a')\right]}$$
$$\geq\frac{\mathbb{E}\left[M(a',q-\tilde{r})\right]}{-\mathbb{E}\left[X(a')\right]}.$$

where the second line follows from a direct corollary of Lemma B.4 $\left(\frac{\partial}{\partial b}\mathbb{E}\left[\psi(\tilde{r})\mid\tau_\iota(q)=\tilde{r},\iota(q)>b\right]<1\right)$, combined with the direct observation that $-\mathbb{E}\left[X(a')\right]>0$ and $\mathbb{E}\left[M(a',q-\tilde{r})\right]>0$. Rearranging this, we have that the condition is equivalent to:

$$\mathbb{P}\left(\tau(b)\in d\tilde{r}\mid m_b^*(\psi)=\tilde{r}\right)\mathbb{E}[X(a')]+\mathbb{P}\left(\tau_\iota(q)\in d\tilde{r},\iota(q)>b\mid m_b^*(\psi)=\tilde{r}\right)\mathbb{E}[M(a',q-\tilde{r})]\leq 0$$

The condition is equivalent to having the expected outcome of the deviation being closer to the unique maximizer of $u_R(\cdot)$, which is a necessary condition for the supposed equation (B.4) to hold under any weakly-concave utility function, as discussed in the proof of Lemma 4. ■

C Distributions of Random Variables of Brownian Motion

Joint Distribution of $X(a)$ and $\iota(q)$

We are interested in studying the distribution $\mathbb{P}(X(a)\in dx,\iota(a)\geq -y)$ for $x\in\mathbb{R}$, $y,a\in\mathbb{R}_+$ and $x\geq -y$. In order to simplify the exposition, we start with an auxiliary Brownian motion with $\sigma=1$ and drift $\mu_Z\in\mathbb{R}$ i.e. $Z(a)=W(a)+\mu_Z a$. We similarly define $\iota(a)$ for this process as $\iota_Z(a)=\inf\{Z(u)\mid u\in[0,a]\}$. Jeanblanc et al. (2009, p. 146) shows the joint distribution of $Z(a)$ and the running minimum $\iota_Z(a)=\min\{z\mid z\in\min_{a'\in[0,a]}Z(a')\}$ as:

$$\mathbb{P}(Z(a)\geq x,\iota_Z(a)\geq -y)=\Phi\left(\frac{-x+\mu_Z a}{\sqrt{a}}\right)-\exp(-2\mu_Z y)\Phi\left(\frac{-x-2y+\mu_Z a}{\sqrt{a}}\right).\tag{C.1}$$

Set $\mu_Z=\frac{\mu}{\sigma}$, then $\sigma Z(a)=X(a)$ and $\sigma\iota_Z(a)=\iota(a)$.[5] It follows that:

$$\mathbb{P}(X(a)\geq x,\iota(a)\geq -y)=\mathbb{P}(\sigma Z(a)\geq x,\sigma\iota_Z(a)\geq -y)=\mathbb{P}\left(Z(a)\geq\frac{x}{\sigma},\iota_Z(a)\geq\frac{-y}{\sigma}\right).$$

So, equation (C.1) generalizes to $X(a)$ as:

$$\mathbb{P}(X(a)\geq x,\iota(a)\geq -y)=\left[\Phi\left(\frac{-x+\mu a}{\sigma\sqrt{a}}\right)-\exp\left(-\frac{2\mu y}{\sigma^2}\right)\Phi\left(\frac{-x-2y+\mu a}{\sigma\sqrt{a}}\right)\right].\tag{C.2}$$

[5] The first claim follows from: $\sigma Z(a)=\sigma(\frac{\mu}{\sigma}a+W(a))=\mu a+\sigma W(a)=X(a)$, and the second claim follows from $\sigma m_Z(a)=\inf\{\sigma Z(u)\mid u\in[0,a]\}=\inf\{\sigma\frac{X(u)}{\sigma}\mid u\in[0,a]\}=\inf\{X(u)\mid u\in[0,a]\}=\iota(a)$.

Using the generalized equation in (C.2), the *density* for $X(a)$ can be obtained by differentiating it with respect to x:

$$\mathbb{P}(X(a) \in dx, \iota(a) \geq -y) = \frac{\phi\left(\frac{-x+\mu a}{\sigma\sqrt{a}}\right) - \exp\left(-\frac{2\mu y}{\sigma^2}\right)\phi\left(\frac{-x-2y+\mu a}{\sigma\sqrt{a}}\right)}{\sigma\sqrt{a}} dx. \tag{C.3}$$

Density and Cumulative Distribution of $\iota(a)$

Our results in the previous section makes it very easy to characterize $\mathbb{P}(\iota(a) \geq -y)$. It follows that $\mathbb{P}(X(a) \geq -y, \iota(a) \geq -y) = \mathbb{P}(\iota(a) \geq -y)$ since $X(a) \geq \iota(a)$. So, $\mathbb{P}(\iota(a) \geq -y)$ can be obtained by setting $x = -y$ in Equation (C.2):

$$\mathbb{P}(\iota(a) \geq -y) = \Phi\left(\frac{\mu a + y}{\sigma\sqrt{a}}\right) - \exp\left(-\frac{2\mu y}{\sigma^2}\right)\Phi\left(\frac{\mu a - y}{\sigma\sqrt{a}}\right) \quad \forall y \in \mathbb{R}_+. \tag{C.4}$$

The density $\mathbb{P}(\iota(a) \in d(-y))$ is given by differentiating the equation (C.4):

$$\mathbb{P}(\iota(a) \in d(-y)) = \frac{2}{\sigma}\left(\frac{\mu}{\sigma}\exp\left(\frac{-2\mu y}{\sigma^2}\right)\Phi\left(\frac{\mu a - y}{\sigma\sqrt{a}}\right) + \frac{1}{\sqrt{a}}\phi\left(\frac{-y-\mu a}{\sigma\sqrt{a}}\right)\right) dy \quad \forall y \in \mathbb{R}_+. \tag{C.5}$$

Distribution of $\tau(-x)$

We now shift our focus from the infimum of Brownian motion to the distribution of hitting times of $-x$ and $\tau(-x)$. An important observation is to note that $\mathbb{P}(\tau(-y) \leq a) = \mathbb{P}(\iota(a) \leq -y)$. This is a very important step to relate our results on $\iota(a) \leq -x$ to $\tau(-x) \leq a$ (Harrison, 2013; Shreve, 2004). This implies that the equation (C.4) for CDF of $\iota(a)$ is identical to the CDF of $\tau(-x)$:

$$\begin{aligned}
\mathbb{P}(\tau(-x) \leq a) &= \mathbb{P}(\iota(a) \leq -x) = 1 - \mathbb{P}(\iota(a) \geq -x) && \forall x \in \mathbb{R}_+ \\
&= 1 - \Phi\left(\frac{\mu a + x}{\sigma\sqrt{a}}\right) + \exp\left(-\frac{2\mu x}{\sigma^2}\right)\Phi\left(\frac{\mu a - x}{\sigma\sqrt{a}}\right) && \forall x \in \mathbb{R}_+ \\
&= \Phi\left(\frac{-\mu a - x}{\sigma\sqrt{a}}\right) + \exp\left(-\frac{2\mu x}{\sigma^2}\right)\Phi\left(\frac{\mu a - x}{\sigma\sqrt{a}}\right) && \forall x \in \mathbb{R}_+.
\end{aligned} \tag{C.6}$$

The density of first hitting time of $-x$ can be obtained from its cumulative distribution given in equation (C.6) by differentiation:

$$\mathbb{P}(\tau(-x) \in da) = \frac{x}{\sigma a\sqrt{a}}\phi\left(\frac{-x-\mu a}{\sigma\sqrt{a}}\right) da \quad \forall x \in \mathbb{R}_+. \tag{C.7}$$

This is the famous distribution of the first hitting time that appears frequently in the study of Brownian motion (Karatzas and Shreve, 2012; Harrison, 2013).

Joint Distribution of $\tau(-x)$ and $\iota(q)$

Our goal this section is to characterize the joint density $\mathbb{P}(\tau(-x) \in da, \iota(q) \in d(-y))$. Establishing this result extends the result by Shepp (1979), which characterizes the joint density of the maximum, its location and the endpoint. We start by studying the cumulative distribution $\mathbb{P}(\tau(-x) \leq a, \iota(q) \leq -y)$. Recall that the maximal decrease is defined as: $\delta(q_1, q_2) = \inf\{X(q_2) - X(q_1) \mid a \in [q_1, q_2]\}$. We can write this expression as:

$$\begin{aligned}
\mathbb{P}(\tau(-x) \leq a, \iota(q) \leq -y) &= \int_0^a \mathbb{P}(\tau(-x) \in du, \delta(u,q) \leq x - y) du && \forall x, y \in \mathbb{R}_+ \\
&= \int_0^a \mathbb{P}(\tau(-x) \in du, \iota(q-u) \leq x - y) du && \forall x, y \in \mathbb{R}_+ \\
&= \int_0^a \mathbb{P}(\tau(-x) \in du) \mathbb{P}(\iota(q-u) \leq x - y) du && \forall x, y \in \mathbb{R}_+. \qquad \text{(C.8)}
\end{aligned}$$

The first line follows from definitions of $\tau(\cdot)$ and $\delta(\cdot)$. The second line follows from the equivalence of $\delta(u, q)$ and $\iota(q - u)$. Finally, the last line follows from the stationary independent increments.

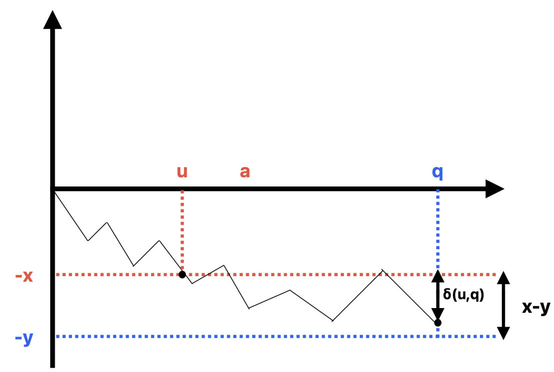

Figure 1: Logic of Equation (C.8). $\mathbb{P}(\tau(-x) \leq a, \iota(q) \leq -y)$ is obtained via integrating over u from 0 to a, for the type of paths illustrated in the picture.

Differentiating equation (C.8), we can obtain:

$$\begin{aligned}
\mathbb{P}(\tau(-x) \in da, \iota(q) \in d(-y)) &= \frac{\partial \mathbb{P}(\tau(-x) \leq a, \iota(q) \leq -y)}{\partial a \, \partial(-y)} && \forall x, y \in \mathbb{R}_+ \\
&= \frac{\partial}{\partial a \, \partial(-y)} \int_0^a \mathbb{P}(\tau(-x) \in du) \mathbb{P}(\iota(q-u) \leq x - y) du && \forall x, y \in \mathbb{R}_+.
\end{aligned}$$

By the fundamental theorem of calculus and the Leibniz rule for differentiation of integrated functions, this expression can be written as:

$$\mathbb{P}(\tau(-x) \in da, \iota(q) \in d(-y)) = \mathbb{P}(\tau(-x) \in da) \mathbb{P}(\iota(q-a) \in d(x-y)) \quad \forall x, y \in \mathbb{R}_+.$$

This expression can be written explicitly using equation (C.5) for $\mathbb{P}(\tau(-x) \in da)$ and (C.7) for

$\mathbb{P}(\iota(q-a) \in d(x-y))$ to get $\mathbb{P}(\tau(-x) \in da, \iota(q) \in d(-y))$ as

$$\frac{x}{\sigma a \sqrt{a}} \phi \left(\frac{-x - \mu a}{\sigma \sqrt{a}} \right) 2 \left(\frac{\mu}{\sigma^2} \exp \left(\frac{2\mu(x-y)}{\sigma^2} \right) \Phi \left(\frac{\mu(q-a) + (x-y)}{\sigma \sqrt{q-a}} \right) + \frac{\phi \left(\frac{(x-y) - \mu(q-a)}{\sigma \sqrt{q-a}} \right)}{\sigma \sqrt{q-a}} \right) dady \ \forall x, y \in \mathbb{R}_+. \tag{C.9}$$

Likelihood Ratio of $\tau(-x)$ and $\tau_\iota(q)$ at a

By definition of $m^*(X)$, the event $m^*(X) = a$ is given by the set of paths $X(\cdot)$ that are in the event $E = \{X(\cdot) \in \mathcal{X} \mid \tau(-x) = a\} \cup \{X(\cdot) \in \mathcal{X} \mid \iota(q) = X(a) > -x\}$. We are interested in the conditional probability that $\mathbb{P}(\tau(-x) \in da \mid \psi(\cdot) \in E) = \mathbb{P}(\tau(-x) \in da \mid m^*(X) = a)$, which is given by:

$$\begin{aligned}
\mathbb{P}(\tau(-x) \in da \mid X(\cdot) \in E) &= \frac{\mathbb{P}(\tau(-x) \in da)}{\mathbb{P}(\tau(-x) \in da) + \mathbb{P}(\tau_\iota(q) \in da \mid \iota(q) > -x)} \quad \forall x \in \mathbb{R}_+ \\
&= \frac{\mathbb{P}(\tau(-x) \in da)}{\mathbb{P}(\tau(-x) \in da) + \int_{-x}^{0} \mathbb{P}(\tau(z) \in da, \iota(q-a) \in dz) dz} \quad \forall x \in \mathbb{R}_+ \\
\mathbb{P}(\tau(-x) \in da \mid X(\cdot) \in E) &= \frac{\mathbb{P}(\tau(-x) \in da)}{\mathbb{P}(\tau(-x) \in da) + \int_{-x}^{0} \mathbb{P}(\tau(z) \in da) \mathbb{P}(\iota(q-a) \in dz) dz} \quad \forall x \in \mathbb{R}_+.
\end{aligned} \tag{C.10}$$

The first line follows from the Bayes Rule and the the definition of E. The second line is the critical observation that $\mathbb{P}(\tau_\iota(q) \in da, \iota(q) > -x) = \int_{-x}^{0} \mathbb{P}(\tau(z) \in da, \iota(q) \in dz) dz$. Finally, the last line follows from stationary independent increments of the Brownian motion.

The analytical expression of the conditional expectation follows from the expressions of $\mathbb{P}(\tau(-x) \in da)$ in equation (C.7) and similarly $\mathbb{P}(\tau(-x) \in da, \iota(q) \in d(-y))$ in equation (C.9). Using this, we can write down $\mathbb{P}(\tau(-x) \in da \mid E)$ in (C.10) explicitly as:

$$\mathbb{P}(\tau(-x) \in da \mid X(\cdot) \in E) = \frac{\frac{x}{\sigma a \sqrt{a}} \phi \left(\frac{-x - \mu a}{\sigma \sqrt{a}} \right)}{\frac{x}{\sigma a \sqrt{a}} \phi \left(\frac{-x - \mu a}{\sigma \sqrt{a}} \right) + 2 \left(\frac{\mu \Phi \left(\frac{\mu(q-a)}{\sigma \sqrt{q-a}} \right)}{\sigma^2} + \frac{\phi \left(\frac{-\mu(q-a)}{\sigma \sqrt{q-a}} \right)}{\sigma \sqrt{q-a}} \right) \int_{-x}^{0} \frac{-z}{\sigma a \sqrt{a}} \phi \left(\frac{z - \mu a}{\sigma \sqrt{a}} \right) dz} da. \tag{C.11}$$

D Distributions of Random Variables of Brownian Meander

The Brownian meander is extensively studied in the theoretical probability literature starting with Durrett et al. (1977). The literature generally focuses on the Brownian meander without the drift, except for the recent paper by Iafrate and Orsingher (2020). They characterize the distribution of the Brownian Meander with a drift, although still fixing the diffusion term to $\sigma = 1$. We extend their results by allowing $\sigma \in \mathbb{R}_+$.

In order to describe the probability law of the Brownian meander, we characterize $\mathbb{P}(X(a) \in dx \mid \iota(q) \geq 0)$ by studying $\mathbb{P}(X(a) \in dx \mid \iota(q) \geq -y)$ for $y \in \mathbb{R}_+$ and taking the limit as $-y \to 0^-$. The reader is referred to Durrett et al. (1977) and Iafrate and Orsingher (2020) for the details of the weak convergence. We take $x, y \in \mathbb{R}_+$, and describe $\mathbb{P}(X(a) \in dx \mid \iota(q) \geq -y)$ using the Bayes' rule and the stationary independent increments property of the Brownian motion $X(a)$:

$$\begin{aligned}\mathbb{P}(X(a)\in dx \mid \iota(q)\geq -y) &= \frac{\mathbb{P}(X(a)\in dx, \iota(q)\geq -y)}{\mathbb{P}(\iota(q)\geq -y)}\\ &= \frac{\mathbb{P}(X(a)\in dx, \iota(a)\geq -y, \iota(q-a)\geq -(x+y))}{\mathbb{P}(\iota(q)\geq -y)}\\ \mathbb{P}(X(a)\in dx \mid \iota(q)\geq -y) &= \frac{\mathbb{P}(X(a)\in dx, \iota(a)\geq -y)\mathbb{P}(\iota(q-a)\geq -(x+y))}{\mathbb{P}(\iota(q)\geq -y)}. \qquad \text{(D.1)}\end{aligned}$$

In the following sections, we will give an analytic expression for equation (D.1) and obtain a companion result when $X(a)$ is replaced with a Brownian bridge.

Distribution of $M(a,q)$

We have characterized the distribution for $M(a,q)$ in equation (D.1) in terms of $\mathbb{P}(X(a)\in dx, \iota(a)\geq -y)$ and $\mathbb{P}(\iota(q)\geq -y)$ given in equation (C.3) and (C.4). Using this, we can write $\mathbb{P}(X(a)\in dx \mid \iota(q)\geq -y)$ as:

$$\frac{\phi\left(\frac{-x+\mu a}{\sigma\sqrt{a}}\right)-\exp\left(-\frac{2\mu y}{\sigma^2}\right)\phi\left(\frac{-x-2y+\mu a}{\sigma\sqrt{a}}\right)}{\Phi\left(\frac{y+\mu q}{\sigma\sqrt{q}}\right)-\exp\left(-\frac{2\mu y}{\sigma^2}\right)\Phi\left(\frac{-y+\mu q}{\sigma\sqrt{q}}\right)}\frac{\left[\Phi\left(\frac{x+y+\mu(q-a)}{\sigma\sqrt{q-a}}\right)-\exp\left(-\frac{2\mu(x+y)}{\sigma^2}\right)\Phi\left(\frac{-x-y+\mu(q-a)}{\sigma\sqrt{q-a}}\right)\right]}{\sigma\sqrt{a}}dx.$$

We are interested in $\lim_{-y\to 0^-}\mathbb{P}(X(a)\in dx \mid \iota(q)\geq -y)$ in order to characterize the distribution of the Brownian meander. We evaluate this limit by looking at the limit of each term. First we have:

$$\lim_{-y\to 0^-}\frac{\Phi\left(\frac{x+y+\mu(q-a)}{\sigma\sqrt{q-a}}\right)-\exp\left(-\frac{2\mu(x+y)}{\sigma^2}\right)\Phi\left(\frac{-x-y+\mu(q-a)}{\sigma\sqrt{q-a}}\right)}{\sigma\sqrt{a}}=\frac{\Phi\left(\frac{x+\mu(q-a)}{\sigma\sqrt{q-a}}\right)-\exp\left(-\frac{2\mu(x)}{\sigma^2}\right)\Phi\left(\frac{-x+\mu(q-a)}{\sigma\sqrt{q-a}}\right)}{\sigma\sqrt{a}}.$$

This can be directly obtained by evaluating it at $-y=0$, so we can then rewrite $\lim_{-y\to 0^-}\mathbb{P}(X(a)\in dx \mid \iota(q)\geq -y)$ as:

$$\frac{\Phi\left(\frac{x+\mu(q-a)}{\sigma\sqrt{q-a}}\right)-\exp\left(-\frac{2\mu(x)}{\sigma^2}\right)\Phi\left(\frac{-x+\mu(q-a)}{\sigma\sqrt{q-a}}\right)}{\sigma\sqrt{a}}\lim_{-y\to 0^-}\frac{\phi\left(\frac{-x+\mu a}{\sigma\sqrt{a}}\right)-\exp\left(-\frac{2\mu y}{\sigma^2}\right)\phi\left(\frac{-x-2y+\mu a}{\sigma\sqrt{a}}\right)}{\Phi\left(\frac{y+\mu q}{\sigma\sqrt{q}}\right)-\exp\left(-\frac{2\mu y}{\sigma^2}\right)\Phi\left(\frac{-y+\mu q}{\sigma\sqrt{q}}\right)}.$$

Focusing on the second part with the limit, we have that:

$$\lim_{-y\to 0^-}\frac{\phi\left(\frac{-x+\mu a}{\sigma\sqrt{a}}\right)-\exp\left(-\frac{2\mu y}{\sigma^2}\right)\phi\left(\frac{-x-2y+\mu a}{\sigma\sqrt{a}}\right)}{\Phi\left(\frac{y+\mu q}{\sigma\sqrt{q}}\right)-\exp\left(-\frac{2\mu y}{\sigma^2}\right)\Phi\left(\frac{-y+\mu q}{\sigma\sqrt{q}}\right)}dx=\frac{\sqrt{q}x\exp\left(\frac{\mu^2 q}{2\sigma^2}\right)\phi\left(\frac{\mu a-x}{\sigma\sqrt{a}}\right)}{a\left(\mu\sqrt{q}\exp\left(\frac{\mu^2 q}{2\sigma^2}\right)\Phi(\frac{\mu\sqrt{q}}{\sigma})+\frac{\sigma}{\sqrt{2\pi}}\right)}dx.$$

Hence, the distribution of the Brownian meander $M(a,q)$ is given by:

$$\mathbb{P}(M(a,q)\in dx)=\frac{\sqrt{q}x}{\sigma a\sqrt{a}}\frac{\exp\left(\frac{\mu^2 q}{2\sigma^2}\right)\phi\left(\frac{\mu a-x}{\sigma\sqrt{a}}\right)\left(\Phi\left(\frac{x+\mu(q-a)}{\sigma\sqrt{q-a}}\right)-\exp\left(-\frac{2\mu x}{\sigma^2}\right)\Phi\left(\frac{-x+\mu(q-a)}{\sigma\sqrt{q-a}}\right)\right)}{\left(\mu\sqrt{q}\exp\left(\frac{\mu^2 q}{2\sigma^2}\right)\Phi(\frac{\mu\sqrt{q}}{\sigma})+\frac{\sigma}{\sqrt{2\pi}}\right)}dx. \quad \text{(D.2)}$$

This distribution coincides with equation (1.4) in Iafrate and Orsingher (2020) when $\sigma=1$, and it coincides with the well-known Rayleigh distribution whenever $\mu=0$, $\sigma=1$ and $a=q$.

Distribution of $M(a,q)$ given $M(q,q)=c$

We now turn our focus to the distribution of $M(a,q)$ given a terminal value $M(q,q)=c$. The special case of $\mu=0$ and $\sigma=1$ is analyzed in Devroye (2010) and Riedel (2021). Our analysis follows a very similar path to Riedel (2021), building on his characterization of the moments of Brownian meander given a terminal value c, in a special case of ours.

Note that when conditioned on $X(q)=c$, the distribution of the Brownian motion is independent of its drift μ. We do the rest of this section assuming $\mu=0$. For this special case, $\mathbb{P}(X(q)\in dc,\iota(q)\in d(-y))$ is known (Karatzas and Shreve, 2012, p. 95):

$$\mathbb{P}(X(q)\in dc,\iota(q)\in d(-y))=\frac{2(c+2y)}{\sqrt{2\pi}\sigma^3\sqrt{q^3}}\exp\left(-\frac{(c+2y)^2}{2\sigma^2 q}\right)dcdy. \quad \text{(D.3)}$$

As the Brownian motion has normally distributed increments, we can write:

$$\mathbb{P}(X(q)\in dc)=\frac{1}{\sqrt{2\pi}\sigma\sqrt{q}}\exp\left(-\frac{c^2}{2\sigma^2 q}\right)dc. \quad \text{(D.4)}$$

We use the Bayes' Rule and equations (D.3) and (D.4) to describe the density of $\iota(q)$ of a Brownian Bridge with terminal value c at q:

$$\mathbb{P}(\iota(q)\in d(-y)\mid X(q)=c))=\frac{\mathbb{P}(\iota(q)\in d(-y),X(q)\in dc))}{\mathbb{P}(X(q)\in dc)}=\frac{2(c+2y)}{\sigma^2 q}\exp\left(\frac{c^2-(c+2y)^2}{2\sigma^2 q}\right)dy.$$

We are interested in $\mathbb{P}(M(a,q)\in dx\mid M(q,q)=c)=\lim_{-y\to 0^-}\mathbb{P}(X(a)\in dx\mid X(q)=c,\iota(q)\geq -y)$. Following a similar line of argumentation as before, we conclude that:

$$\frac{\mathbb{P}(X(a)\in dx,\iota(q)\geq -y,X(q)=c)}{\mathbb{P}(\iota(q)\geq -y,X(q)=c)}=\frac{\mathbb{P}(X(a)\in dx,\iota(q)\geq -y\mid X(q)=c)}{\mathbb{P}(\iota(q)\geq -y\mid X(q)=c)}$$

$$=\frac{\mathbb{P}(X(a)\in dx,\iota(a)\geq -y\mid X(q)=c)\mathbb{P}(\delta(a,q)\geq -x\mid X(q-a)=c-x)}{\mathbb{P}(\iota(q)\geq -y\mid X(q)=c)}$$
$$=\frac{\mathbb{P}(X(a)\in dx\mid X(q)=c)\mathbb{P}(\iota(a)\geq -y\mid X(a)=x,X(q)=c)\mathbb{P}(\delta(a,q)\geq -x,X(q-a)=c-x)}{\mathbb{P}(\iota(q)\geq -y,X(q)=c)}. \quad \text{(D.5)}$$

The first equality follows from Bayes rule. The second and third equality follow from stationary independent increments and the definitions of the related random variables. In order to give a closed-form expression for equation (D.5), we study each term separately. First, we have $\mathbb{P}(X(a) = x \mid X(q) = c)$ in the numerator. This is just the well-known distribution over a Brownian bridge given by (Shreve, 2004; Harrison, 2013):

$$\mathbb{P}(X(a) \in dx \mid X(q) = c) = \frac{1}{\sigma\sqrt{\frac{a}{q}}\sqrt{(q-a)}}\phi\left(\frac{x - \frac{ca}{q}}{\sigma\sqrt{\frac{a}{q}}\sqrt{q-a}}\right) dx. \tag{D.6}$$

We shift our focus to the second expression on the numerator. Note that $\mathbb{P}(\iota(a) \geq -y \mid X(a) = x, X(q) = c) = \mathbb{P}(\iota(a) \geq -y \mid X(a) = x)$. Thus, this expression can be directly calculated from (D.3).

$$\mathbb{P}(\iota(a) \geq -y \mid X(a) = x) = \int_{-y}^{0} \mathbb{P}(X(a) \in dx, \iota(a) \in du)du = 1 - \exp\left(\frac{-2y(x+y)}{\sigma^2 a}\right). \tag{D.7}$$

$\mathbb{P}(\delta(a,q) \geq -x \mid X(q-a) = c-x)$ and $\mathbb{P}(\iota(a) \geq -y \mid X(q) \in dc)$ can be obtained with similarly.

$$\begin{aligned}\mathbb{P}(\delta(a,q) \geq -x \mid X(q-a) = c-x) &= \mathbb{P}(\iota(q-a) \geq -x \mid X(q-a) = c-x) \\ &= 1 - \exp\left(\frac{-2x(c-x+x)}{\sigma^2(q-a)}\right) \\ \mathbb{P}(\delta(a,q) \geq -x \mid X(q-a) = c-x) &= 1 - \exp\left(\frac{-2cx}{\sigma^2(q-a)}\right). &\text{(D.8)} \\ \mathbb{P}(\iota(q) \geq -y \mid X(q) = c) &= 1 - \exp\left(\frac{-2y(c+y)}{\sigma^2 q}\right). &\text{(D.9)}\end{aligned}$$

We are interested in $\mathbb{P}(X(a) = x \mid X(q) = c, \iota(q) \geq -y)$ described in (D.5) as $-y \to 0^-$. Observe that $\mathbb{P}(\iota(q-a) \geq -x \mid X(q-a) \in d(c-x))$ and $\mathbb{P}(\iota(q-a) \geq -x \mid X(q-a) \in d(c-x))$ does not depend on y. Inspecting the limit involving equation (D.7) and (D.9), we have that:

$$\lim_{y \to 0^-} = \frac{1 - \exp(\frac{2y(x+y)}{\sigma^2 t})}{1 - \exp(\frac{2y(c+y)}{\sigma^2 q})} = \frac{qx}{ca}. \tag{D.10}$$

Using (D.5), (D.6), (D.8), and (D.10) we can write $\mathbb{P}(X(a) = x \mid X(q) = c, \iota(q) \geq -y)$ as:

$$\mathbb{P}(X(a) \in dx \mid X(q) \in dc, \iota(q) \geq 0) = \frac{qx}{ca}\frac{\phi\left(\frac{x - \frac{ca}{q}}{\sqrt{\frac{a}{q}}\sqrt{q-a}\sigma}\right)}{\sqrt{\frac{a}{q}}\sqrt{(q-a)}\sigma}\left(1 - \exp\left(\frac{-2cx}{\sigma^2(q-a)}\right)\right) dx. \tag{D.11}$$

So, we can rewrite Equation (D.11) as the following. We make two observations to simplify the expression. First

$$\frac{qx}{ca}\frac{1}{\sqrt{\frac{a}{q}}\sqrt{q-a}\sigma}=\frac{xq\sqrt{q}}{ca\sqrt{a}\sqrt{q-a}\sigma}.$$

Second, by the definition of $\psi(\cdot)$, we have

$$\left(1-\exp\left(\frac{-2cx}{\sigma^2(q-a)}\right)\right)\phi\left(\frac{x-\frac{ca}{q}}{\sqrt{\frac{a}{q}}\sqrt{q-a}\sigma}\right)=\phi\left(\frac{x-\frac{ca}{q}}{\sqrt{\frac{a}{q}}\sqrt{q-a}\sigma}\right)-\phi\left(\frac{x+\frac{ca}{q}}{\sqrt{\frac{a}{q}}\sqrt{q-a}\sigma}\right).$$

Thus, we have that:

$$\mathbb{P}(X(a)\in dx\mid X(q)=c,\iota(q)\geq 0)=\frac{xq\sqrt{q}}{ca\sqrt{a}\sqrt{q-a}\sigma}\left[\phi\left(\frac{x-\frac{ca}{q}}{\sqrt{\frac{a}{q}}\sqrt{q-a}\sigma}\right)-\phi\left(\frac{x+\frac{ca}{q}}{\sqrt{\frac{a}{q}}\sqrt{q-a}\sigma}\right)\right]dx. \quad \text{(D.12)}$$

This expression coincides with equation (3.4) in Riedel (2021) whenever $\sigma=1$, and previously studied by Durrett et al. (1977).

Moments of $M(a,q)$ conditional on $M(q,q)=c$

Using equation (D.12), we can write the expectation:

$$\mathbb{E}[X(a)\mid X(q)=c,\iota(q)\geq 0]=\int_0^\infty x\mathbb{P}(X(a)\in dx\mid X(q)\in dc,\iota(q)\geq 0)dx.$$

This integral is very similar to the one studied by Riedel (2021) in Equation (9.1). We follow their approach for solving it. First, we can simplify the integral by writing the normal density explicitly and taking all possible constants out. We write $\mathbb{E}[X(a)\mid X(q)\in dc,\iota(q)\geq 0]$ as:

$$\frac{q\sqrt{q}}{ca\sqrt{a}\sqrt{q-a}\sigma}\frac{1}{\sqrt{2\pi}}\exp\left(\frac{-c^2a}{2q(q-a)\sigma}\right)\int_0^\infty x\exp\left(\frac{-qx^2}{2a(q-a)\sigma^2}\right)\left(\exp\left(\frac{xc}{(q-a)\sigma^2}\right)-\exp\left(\frac{-xc}{(q-a)\sigma^2}\right)\right)dx.$$

We set $A=\frac{q}{2a(q-a)\sigma^2}$ and $B=\frac{c}{(q-a)\sigma^2}$, and simplify the expression as:

$$\mathbb{E}[X(a)\mid X(q)=c,\iota(q)\geq 0]=\frac{q\sqrt{q}}{ca\sqrt{a}\sqrt{q-a}\sigma}\frac{1}{\sqrt{2\pi}}\exp\left(\frac{-c^2a}{2q(q-a)\sigma^2}\right)\int_0^\infty x^2\exp\left(-Ax^2\right)\sinh(Bx)dx. \quad \text{(D.13)}$$

Here $\sinh(\cdot):=e^x-e^{-x}$ denotes the Hyperbolic Sine function. This integral coincides with Equation (9.3) in Riedel (2021), except that the coefficients A and B are different. Riedel (2021) shows that for constants A and B, we have that:

$$\int_0^\infty x^2\exp\left(-Ax^2\right)\sinh(Bx)dx=\left[\frac{\sqrt{\pi}\left[2A+B^2\right]e^{\frac{b^2}{4A}}\operatorname{erf}\left(\frac{B}{2\sqrt{A}}\right)}{4A^2\sqrt{A}}+\frac{B}{2A^2}\right].$$

So, $\mathbb{E}[X(a) \mid X(q) = c, \iota(q) \geq 0]$ given by equation (D.13) is:

$$\mathbb{E}[X(a) \mid X(q) = c, \iota(q) \geq 0] = \frac{q\sqrt{q}}{ca\sqrt{a}\sqrt{q-a}\sigma} \frac{2}{\sqrt{2\pi}} \exp\left(\frac{-c^2 a}{2q(q-a)\sigma^2}\right) \left[\frac{\sqrt{\pi}\left[2A + B^2\right] e^{\frac{B^2}{4A}} \operatorname{erf}\left(\frac{B}{2\sqrt{A}}\right)}{4A^2\sqrt{A}} + \frac{B}{2A^2}\right].$$

We can simplify this expression as:

$$\mathbb{E}[X(a) \mid X(q) = c, \iota(q) \geq 0] = \frac{\sigma^2(q-a) + \frac{c^2 a}{q}}{c} \operatorname{erf}\left[\frac{c\sqrt{a}}{\sigma\sqrt{2q(q-a)}}\right] + \exp\left(\frac{-c^2 a}{2q(q-a)\sigma^2}\right)\sqrt{\frac{2a(q-a)}{q\pi}}\sigma, \quad \text{(D.14)}$$

from the following identities involving $A = \dfrac{q}{2a(q-a)\sigma^2}$ and $B = \dfrac{c}{(q-a)\sigma^2}$:

$$2A + B^2 = \frac{\sigma(q-a)q + c^2 a}{a(q-a)^2\sigma^4}, \qquad 4A^2\sqrt{A} = \frac{\sqrt{2}(q^2\sqrt{q})}{\sigma^5(a^2\sqrt{a})(q-a)^2\sqrt{q-a}},$$

$$\frac{B}{2\sqrt{A}} = \frac{c\sqrt{a}}{\sqrt{2(q-a)q}\sigma}, \qquad \frac{B^2}{4A} = \frac{c^2 a}{2q(q-a)\sigma^2}, \qquad \frac{B}{2A^2} = \frac{2\sigma^2(q-a)a^2 c}{q}.$$

E Proofs of Corollaries in the Main Text

Proof of Corollary 1. Consider the first-point strategies $m^*_q(\cdot)$ and $m^*_{q'}(\cdot)$ for games with action spaces $\mathcal{A} = [0, q]$ and $\mathcal{A}' = [0, q']$ with $q < q' \leq q_{\max}$. By definition of a first-point strategy, for every $\psi \in \Psi$ it holds that $\psi(m^*_q(\psi)) \geq \psi(m^*_{q'}(\psi)) \geq b$, where the set of paths that this holds with equality has measure zero under $\omega(\cdot)$. The conclusion follows immediately. ■

Proof of Corollary 2. Similarly, take any realized path $\psi(\cdot)$. Consider the first-point strategies $m^*_b(\cdot)$ and $m^*_{b'}(\cdot)$ for games with bias $b < b'$ such that the first-point equilibrium exists. By definition of a first-point strategy and continuity of the Brownian path:

1. If $\psi(m^*_b(\psi)) = b$, then $\psi(m^*_{b'}(\psi)) = b'$.

2. If $b' > \psi(m^*_b(\psi)) > b$, then $\psi(m^*_{b'}(\psi)) = b'$.

3. If $\psi(m^*_b(\psi)) > b'$, then $\psi(m^*_{b'}(\psi)) = \psi(m^*_{b'}(\psi)) > b'$.

Thus, for all paths $\psi(m^*_{b'}(\psi)) - b' < \psi(m^*_b(\psi)) - b$ with the inequality is strict for a measurable set of paths. The opposite is true relative to the receiver's ideal outcome of 0. The conclusion follows. ■

Proof of Corollary 3. By construction, we have that:

$$\mathbb{E}[\psi(r^*) \mid m^*(\psi) = r^*] = \mathbb{P}(\iota(q) > b)\ \mathbb{E}[\iota(q) \mid \iota(q) > b] + \mathbb{P}(\iota(q) \leq b)\ b$$

We have that $\psi(a) = \mu a + \sigma W(a)$ where $W(a)$ is the Wiener Process. It is a well-established result that $W(a) < 0$ for some a with probability one. Thus, with probability one there is an a such that $\psi(a) < \mu a$. As $\sigma \to \infty$ we have that $\psi(a) < b$ for some $a \in [0, q]$ with probability 1. It follows that $\sigma \to \infty$, $\mathbb{P}(\iota(q) \leq b) \to 1$ and the result holds. ■

F Natural Language Refinements

F.1 Farrell's Neologism-Proof Equilibria

In order to study meaningful communication, Farrell (1993) imposes additional structure on the arbitrary message space. We extend his approach to our richer state spapce, by assuming that for each measurable (with respect to the prior $\omega(\cdot)$) subset of the type space $\Theta \subseteq \Psi$, and for every Perfect Bayesian equilibrium (m, a, ω), there exists a message $n(\Theta) \in \mathcal{M}$ unused in the equilibrium (m, a, ω).[6] Message $n(\Theta)$ has the literal meaning $\psi \in \Theta$ or in words "my type is in Θ".

Thus, messages have a natural meaning about types. This allows the sender to use 'neologisms' – messages not used in the equilibrium with a natural meaning – to communicate about her type without being constrained to the meaning of equilibrium messages. In the words of Farrell (1993), this eliminates the 'comprehensibility' barrier in communication, and 'the credibility' of messages is the only barrier to communication.

Definition 1 *For each Θ, a **neologism** is a message $n(\Theta)$. A **neologism is believed** if it causes the receiver to adopt beliefs by conditioning on the event $\psi \in \Theta$, i.e. $\omega(\psi \mid \psi \in \Theta)$.*

A neologism $n(\Theta)$ is credible if types in Θ prefer the resulting payoff if $n(\Theta)$ is believed over the equilibrium payoff; and everyone else prefers the equilibrium payoff over the resulting payoff if $n(\Theta)$ is believed. Formally, we can state if a neologism is credible based on the following set of inequalities:

Definition 2 *Neologism $n(\Theta)$ is **credible** relative to an equilibrium (m, a, ω) if it satisfies (N1) and (N2):*

(N1) $u^S(a' \mid \psi) > u^S(m, a \mid \psi)$ *for every* $\psi \in \Theta$ *and* $a' \in \arg\max_{a'' \in \Delta A} u^R(a'', \psi \mid \psi \in \Theta)$.

(N2) $u^S(a' \mid \psi) \leq u^S(m, a \mid \psi)$ *for every* $\psi \notin \Theta$ *and* $a' \in \arg\max_{a'' \in \Delta A} u^R(a'', \psi \mid \psi \in \Theta)$.

Farrell (1993) provides the literal meaning of the neologism $n(\Theta)$ as the following

[6] Farrell (1993) and Matthews, Okuno-Fujiwara, and Postlewaite (1991) both assume that the state space is finite to avoid measurability issues. We extend their discussion to our environment with richer state space. We flag certain measurability issues in Remark 2, and discuss how we handle them.

"My type is $\psi \in \Theta$. You should believe me because if my type is in Θ, it is better for me to reveal that my type is in the set Θ instead of obtaining the equilibrium outcomes. If my type is not in Θ, it is better for me to receive the equilibrium outcome instead of you believing that my type is in the set Θ."

Definition 3 *An equilibrium (m, a, ω) is **neologism-proof** if no neologism is credible relative to it. Equilibrium payoff is neologism proof if all the equilibria giving rise to it are neologism proof.*

Remark 1 *Neologism-proofness is essentially a property about equilibrium payoffs. It follows from (N1) and (N2) that an equilibrium is neologism proof if all equilibria giving rise to the same payoffs are neologism proof.*

Remark 2 *In the Brownian model certain types of neologism might lead to undefined posterior beliefs. For example, consider the neologism "The realized path is differentiable." The posterior $\omega(\psi \mid \psi \in \Theta)$ corresponding to this neologism is not well-defined, and can have an arbitrary value. This is not an issue for our conclusions, because the equilibria that we show to be neologism-proof are immune to any neologism and arbitrary beliefs $\omega(\psi \mid \psi \in \Theta)$ as the equilibrium outcomes are sender-optimal.*

F.2 Matthews, Okuno-Fujiwara, and Postlewaite's Announcement-Proof Equilibria

An extension of neologism-proofness of Farrell (1993) is the announcement-proofness notion of Matthews, Okuno-Fujiwara, and Postlewaite (1991). Their insight is that one should consider all possible neologism together, instead of considering deviations by a single neologism in isolation. They formalize this by studying deviations from the equilibrium by an announcement strategy which specify all sender types that use neologisms (deviant types) and which neologisms they use.

Definition 4 *An **announcement strategy** for the sender is (n, Θ) where $\Theta \subseteq \Psi$ is a measurable set of deviant types and $n : \Theta \to \Delta M$ is a neologism strategy. Moreover, the triplet (s, n, Θ) with $s \in n(\Theta)$ is called an **announcement**.*

Definition 5 *The **announcement is believed** if the receiver is convinced that every type $\psi \in \Theta$ makes the announcement (s, n, Θ) with s chosen according to $n(\cdot \mid \psi)$, and no type that is not in Θ would have made such an announcement. If the announcement is believed the receiver has a posterior belief given by the Bayes' Rule, i.e. $\omega(\psi \mid n(\psi) = s, \psi \in \Theta)$.*

Matthews, Okuno-Fujiwara, and Postlewaite (1991) provides the literal meaning of announcement (s, n, Θ) as:

"My type is $\psi \in \Theta$, and I am sending message s according to strategy $n(\cdot \mid \psi)$. If I had been type $\psi' \in \Theta$, I would have made an announcement that differed only in so far as s would have been chosen according to strategy $n(\cdot \mid \psi')$. If my type had not been in Θ, I would not have used announcement strategy (n, Θ)."

In order to formalize which announcements are credible, and thus believed, we introduce some additional definitions from Matthews, Okuno-Fujiwara, and Postlewaite (1991).

- A pessimistic sender of type ψ who announces (s, n, Θ) when it is believed expects to receive:

$$\underline{u}^S(s, n, \Theta \mid \psi) = \min\left\{u^S(a \mid \psi) : a \in \arg\max_{a' \in \Delta\mathcal{A}} \left\{u^R(a' \mid n(\psi) = s, \psi \in \Theta)\right\}\right\}.$$

- An optimistic sender of type ψ who announces (s, n, Θ) when it is believed expects to receive:

$$\overline{u}^S(s, n, \Theta \mid \psi) = \max\left\{u^S(a \mid \psi) : a \in \arg\max_{a' \in \Delta\mathcal{A}} \{u^R(a' \mid n(\psi) = s, \psi \in \Theta)\}\right\}$$

Definition 6 *An announcement strategy (n, Θ), and the corresponding announcements (s, n, Θ) are **weakly credible** at equilibrium (m, a, ω) if they satisfy the following three conditions:*

(A1) Pessimistic deviant types prefer the announcement payoff to the equilibrium, with the inequality strict for some $\psi \in \Theta$ and $s \in n(\psi)$:

$$\underline{u}^S(s, n, \Theta \mid \psi) \geq u^S(m, a \mid \psi) \text{ for every } \psi \in \Theta \text{ and } s \in n(\psi)$$

(A2) Optimistic non-deviant types prefer the equilibrium to the announcement:

$$\overline{u}^S(s, n, \Theta \mid \psi) \leq u^S(m, a \mid \psi) \text{ for every } \psi \in \Psi \setminus \Theta \text{ and } s \in n(\Theta).$$

(A3) The announcement is internally consistent:

$$\underline{u}^S(s, n, \Theta \mid \psi) \geq \overline{u}^S(s', n, \Theta) \text{ for all } \psi \in \Theta,\ s \in n(\psi),\ \text{and } s' \in n(\Theta) \setminus \{s\}.$$

Definition 7 *An equilibrium (m, a, ω) is **strongly announcement-proof** if no announcement is weakly credible relative to (m, a, ω).*

Remark 3 *Strong announcement-proofness is a stronger condition compared to neologism proofness. It is an immediate observation that neologism-proofness is a special case of announcement-proofness when all sender deviant types Θ follow the same strategy $n(\Theta) = \Theta$. This can be formally observed by comparing (N1) and (N2) with (A1) and (A2) – and (A3) is vacuously satisfied. Thus, every **neologism** forms a **weakly credible announcement**, but not vice versa.*

Matthews, Okuno-Fujiwara, and Postlewaite (1991) also introduces a weaker version of strongly announcement-proof equilibria called announcement-proof equilibrium. This is done by strengthening the weak credibility with an additional condition (A4).

Definition 8 *An announcement strategy (n, Θ), and the corresponding announcements (s, n, Θ) are **credible** at equilibrium (m, a, ω) if they satisfy conditions (A1), (A2), (A3) and (A4).*

A4. Consider any other (n', Θ') that satisfies (A1),(A2) and (A3) relative to (m, a, ω), then:

$$\underline{u}^S(s, n, \Theta \mid \psi) \geq \overline{u}^S(s, n, \Theta \mid \psi) \text{ for all } \psi \in \Theta \cap \Theta',\ s \in n(\psi),\ \text{and } s' \in n'(\psi).$$

Definition 9 *An equilibrium (m, a, ω) is **announcement-proof** if no announcement is credible relative to (m, a, ω). Moreover, either all or none of the equilibrium that gives rise to a particular outcome are strongly announcement-proof.*

Matthews, Okuno-Fujiwara, and Postlewaite (1991) introduces an even weaker refinement called weakly announcement-proof equilibria by strengthening credible announcements further by introducing strongly credible announcements. The main purpose of this extension is to address issues regarding the *Stiglitz Critique*.[7] We skip the discussion of these concepts, and refer the interested reader to their detailed discussion.

Remark 4 *It is also true that strong announcement-proofness and announcement-proofness are essentially a property about equilibrium payoffs. Either all or none of the equilibria that gives rise to a particular equilibrium payoff are (strongly) announcement-proof, as conditions A1-A3 refer to the equilibrium only through the interim utility levels it gives the Sender and condition A4 is not a condition on the equilibrium.*

G Other Complex Environments

In this section we provide the formal details for the environments discussed in Section 6.2. Unless amended otherwise, the details of the model are as in the main text.

Minimal Complexity:
Let the state space be $\Psi = \mathbb{R} \times \{-1, 1\}$ such that for $(w, z) \in \Psi$ we have that $\psi(a \mid w, z) = b + z(a - w)$. The receiver has prior beliefs given by $\omega((w, z))$ over Ψ. Note there is no known status quo point. The sender follows the first-point strategy (the optimal action is now unique and the 'first' modifier moot) $m^* : \Psi \to \mathbb{R}$.

[7]The details of the argument credited to Stiglitz can be found in Cho and Kreps (1987). Matthews, Okuno-Fujiwara, and Postlewaite (1991) argues that the critique is not convincing. They explain that the rationale behind the refinement is that the equilibrium (m, a, ω) is expected to be played, which contradicts with the argument of Stiglitz's Critique

For the receiver, the set of states consistent with a recommendation r^* are:

$$m^{*-1}(r^*) = \{(r^*, -1), (r^*, 1)\}.$$

$m^{*-1}(r^*)$ is not single valued for any $r^* \in \mathbb{R}$, thus $m^*(\cdot)$ satisfies partial invertibility. Sender strategy $m^*(\cdot)$ satisfies response uncertainty, as the set of optimal responses to the states that are consistent with message r^* are given by:

$$a(m^{*-1}(r^*)) = \{r^* - b, r^* + b\}$$

To check receiver incentive compatibility, consider a deviation $a' \in \mathcal{A}$ given recommendation r^*. The receiver's conditional beliefs are $\omega((r^*, 1) \mid m(\psi) = r^*)$ and $\omega(r^*, -1 \mid m^*(\psi) = r^*)$ where $\omega(r^*, 1 \mid m^*(\psi) = r^*) + \omega(m^*, -1 \mid m^*(\psi) = r^*) = 1$.

$$\begin{aligned}
\mathbb{E}(a \mid m^*(\psi) = r^*) &= -\omega(m^*, 1 \mid m^*(\psi) = r^*)(b + (a - r^*))^2 - \omega(r^*, -1 \mid m^*(\psi) = r^*)(b - (a - r^*))^2 \\
&= -b^2 - (a - r^*)^2 - 2b(a - r^*)(\omega(r^*, 1 \mid r^*) - 1) \\
\frac{d}{da}\mathbb{E}(a \mid m^*(\psi) = r^*) &= -2 \mid a - r^* \mid -2b(2\omega(r^*, 1 \mid m^*(\psi) = r^*) - 1) \\
\frac{d^2}{da^2}\mathbb{E}(a \mid m^*(\psi) = r^*) &= -2
\end{aligned}$$

The first order condition is satisfied if and only if $\omega(m^*, 1 \mid m^*(\psi) = r^*) = \frac{1}{2}$.

Sender-Receiver Misalignment without Directional Uncertainty:

Let the action and message spaces be the set of positive integers, $\mathbb{Z}_+$. The state space is $\Psi = \mathbb{Z}_+ \cup \{1, 2\}$ such that for $(w, z) \in \Psi$:

$$\psi(a \mid w, z) = \begin{cases} b & \text{if } a = w \\ 0 & \text{if } a = w + z \\ 100b & \text{if } a \notin \ \{w, w + z\} \end{cases}$$

The receiver has beliefs prior belief given by $\omega((w, z))$ over Ψ. Note there is no known status quo point. The sender follows the first-point strategy (the optimal action is now unique and the 'first' modifier moot). $m^* : \Psi \rightarrow \mathbb{R}$.

For the receiver, the set of states consistent with a recommendation r^* are:

$$m^{*-1}(r^*) = \{(r^*, 1), (r^*, 2)\}.$$

$m^{*-1}(r^*)$ is not single valued for any $r^* \in \mathbb{R}$. Thus $m^*(\cdot)$ satisfies partial invertibility. Sender strategy $m^*(\cdot)$ satisfies response uncertainty, as the set of optimal responses to the states that are consistent with recommendation r^* are given by:

$$a(m^{*-1}(r^*)) = \{r^* + 1, r^* + 2\}$$

To check receiver incentive compatibility, consider a deviation $a' \in \mathcal{A}$ given recommendation r^*. The receiver's conditional beliefs are $\omega((r^*, 1) \mid m^*(\psi) = r^*)$ and $\omega((r^*, 2) \mid m^*(\psi) = r^*)$ where $\omega(r^*, 1 \mid m^*(\psi) = r^*) + \omega(r^*, 2 \mid m^*(\psi) = r^*) = 1$.

$$\mathbb{E}(a \mid m^*) = \begin{cases} -b^2 & \text{if } a = r^* \\ -\omega(r^*, 2 \mid m^*(\psi) = r^*)10000b^2 & \text{if } a = r^* + 1 \\ -\omega(r^*, 1 \mid m^*(\psi) = r^*)10000b^2 & \text{if } a = r^* + 2 \\ -10000b^2 & \text{if } a \notin \{r^*, r^* + 1, r^* + 2\} \end{cases}$$

It is optimal for the receiver to follow the recommendation as long as $\omega(r^*, 1 \mid r^*(\psi) = r^*)$ is not too close to 0 or 1.

Local Uncertainty:

The space of outcome maps Ψ is the paths of Ornstein-Uhlenbeck process. Formally, Ψ is the set of solutions to the following stochastic differential equation where $W(a)$ is the Wiener process:

$$d\psi(a) = -\kappa\left(\psi(0) - \psi(a)\right)da + \sigma dW(a)$$

For this process κ is the mean-reversion coefficient, and σ is the constant volatility term. This environment has the same state space as the Brownian environment, differing only in how the states are translated into outcomes via the outcome mappings.

Partial invertibility is satisfied under the first-point strategy as, just like the Brownian Motion, there are infinitely many paths $\psi(\cdot)$ of the Ornstein-Uhlenbeck process consistent with the message $m^*(\psi) = r^*$. Moreover, for every action $a \in \mathbb{R}_{++}$ there exists a realization of ψ such that $\psi(a) \in \arg\max_a -\psi(a)^2$, and the response uncertainty is also satisfied.

Consider the recommendation r^* and a deviation $a \in \mathbb{R}$. Deviations to $a < r^*$ are worse for the receiver as, by the first-point strategy and the continuity of OU process, $\psi(a) > b$ for every $a < r^*$ with certainty. For deviations $a > r^*$, the expected outcome and variance are, recalling that $\psi(0)$ is the mean of the process:

$$\mathbb{E}[\psi(a) \mid m^*(\psi) = r^*] = \psi(0) - \left(\psi(0) - \psi(r^*)\right)\exp(-\kappa(a - r^*))$$

$$\text{Var}(\psi(a) \mid m^*(\psi) = r^*) = \frac{\sigma^2}{2\kappa}(1 - \exp[-2\kappa(a - r^*)])$$

As $\exp(-\kappa(a - r^*)) < 1$, the expected outcome is weakly greater than $m^*(\psi)$ for $a > r^*$ whenever $\psi(r^*) \leq \psi(0)$, which must be true given the first-point strategy. As variance is positive, it is optimal for the receiver to accept the recommendation. (Note that this argument holds even if $\psi(r^*) \in (b, \psi(0)]$).

More Knowledge About the World:
Ψ is the set of all paths of a Brownian Motion with scale σ together with $\psi(0) > b$ and $\psi(q) > b$. The underlying state space is the same as in the Brownian motion (and OU) environments. For the first-point strategy and recommendation r^*, all actions $a < r^*$ are dominated by the recommendation itself. For $a > r^*$, the receiver's beliefs are given by:

$$\mathbb{E}[\psi(a)] = \psi(r^*) + \frac{\psi(q) - \psi(r^*)}{q - r^*}(a - r^*) > \psi(r^*) \text{ as } \psi(q) > \psi(r^*)$$
$$\text{Var}(\psi(a)) = \sigma^2 \frac{(q - a)(a - r^*)}{q - r^*}.$$

The first-point strategy satisfies partial invertibility, response uncertainty, and receiver incentive compatibility are analogously to the Brownian motion and Ornstein-Uhlenbeck environments.

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